3. Quadratic Formula for the Quadratic Equation $ax^2 + bx + c = 0$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression $b^2 - 4ac$ is called **discriminant**. Discriminant is the part of quadratic formula that changes the value of it.

$D > 0$	Quadratic equation has two different real roots
$D = 0$	Quadratic equation has one real root
$D < 0$	Quadratic equation has no real roots

Example: Find all real roots of the quadratic equation $x^2 + 3x - 4 = 0$.

Solution:

$x^2 + 3x - 4 = 0 \Rightarrow a = 1, b = 3, c = -4$

Use quadratic formula:

$$x_{1,2} = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot (-4)}}{2 \cdot 1} = \frac{-3 \pm \sqrt{25}}{2} = \frac{-3 \pm 5}{2} = -4, 1$$

Answer: Two real solutions $x_1 = -4$ and $x_2 = 1$.

Example: For which value of c quadratic equation $2x^2 - 9x + c = 0$ has no real solutions?

Solution:

$2x^2 - 9x + c = 0 \Rightarrow a = 2, b = -9, c = c$

Find the discriminant $(-9)^2 - 4 \cdot 2 \cdot c = 81 - 8c$. If the quadratic equation has no real solutions, then

$$81 - 8c < 0 \Rightarrow c > \frac{81}{8}$$

Answer: Equation has no real solutions for all $c > \frac{81}{8}$.

5. Mean of Consecutive Numbers

When you have a large set of consecutive numbers, or numbers that have a consistent difference, instead of adding all these numbers and dividing them by the amount to find the mean, you could simply add the first number and the last number in the set and divide the sum by 2.

Example: What is the mean of the set of numbers below?

$$3, 6, 9, 12, 15, 18, 21, 24, \ldots, 300, 303, 306, 309, 312$$

Solution:

$$3,\ 6,\ 9,\ 12,\ 15,\ 18,\ 21, \ldots, 300,\ 303,\ 306,\ 309,\ 312$$

$$\underset{+3}{\frown}\ \underset{+3}{\frown}\ \underset{+3}{\frown}\ \underset{+3}{\frown}\ \underset{+3}{\frown}\ \underset{+3}{\frown} \qquad \underset{+3}{\frown}\ \underset{+3}{\frown}\ \underset{+3}{\frown}\ \underset{+3}{\frown}$$

First number $= 3$

Last number $= 312$

$\Rightarrow$ The mean $= \frac{3+312}{2} = \frac{315}{2} = 157.5$

Answer: 157.5.

6. Arithmetic Sequence

An arithmetic sequence is a sequence of numbers such that the difference of any two successive members of the sequence is a constant.

a_n – n^{th} number in the sequence

a_1 – 1st number in the sequence

d – common difference

n – position of number in the sequence

$$a_n = a_1 + (n-1)d$$

Example: What is the 100^{th} number in the sequence 2, 4, 6, 8, 10, 12, 14, 16, 18,...?

Solution: $2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14,\ 16,\ 18\ldots$

$\underset{+2}{\frown}\ \underset{+2}{\frown}\ \underset{+2}{\frown}\ \underset{+2}{\frown}\ \underset{+2}{\frown}\ \underset{+2}{\frown}\ \underset{+2}{\frown}\ \underset{+2}{\frown}$

$a_{100} = 2 + (100-1)\cdot 2 \Rightarrow a_{100} = 2 + 99\cdot 2 = 2 + 198 = 200.$

Answer: 200.

This is because there are four spaces to be filled: _, _, _, _.
The first space can be filled by any one of the four letters. The second space can be filled by any of the remaining three letters. The third space can be filled by any of the two remaining letters and the final space must be filled by the one remaining letter.
The total number of possible arrangements is therefore $4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$.
Answer: In 24 different ways.

Example: How many different 5-digit numbers can you create using all digits 0, 1, 2, 3, 4?

Solution: You can arrange digits 0, 1, 2, 3, 4 in $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$ different ways. But 5-digit number cannot start with 0, so you have subtract the number of 5-digit numbers starting with 0. There are $4! = 24$ such numbers, so you can create $120 - 24 = 96$ different 5-digit numbers.

Answer: 96.

10. Combinations (order doesn't matter)

The number of ways of selecting r objects from n unlike objects is

$$C_r^n = \frac{n!}{r! \cdot (n-r)!}$$

Example: How many different ways can you select two letters from the set of letters: A, B, and C? (Hint: In this problem, order is NOT important; i.e., AB is considered the same selection as BA).

Solution: One way to solve this problem is to list all of the possible selections of two letters from the set of A, B, and C. They are: AB, BC, and AC. Thus, there are 3 possible combinations.
Another approach is to use combinations. We have 3 distinct objects so $n = 3$. And we want to arrange them in groups of 2, so $r = 2$. Thus, the number of combinations is:

$$C_2^3 = \frac{3!}{2! \cdot (3-2)!} = \frac{3!}{2! \cdot 1!} = \frac{1 \cdot 2 \cdot 3}{1 \cdot 2 \cdot 1} = 3.$$

Answer: 3 different ways.

Example: There are 10 teams in the WNBA. They play each other twice. How many matches have taken place?

Solution: We have 10 teams, so $n = 10$. And we want 2 teams to play match, so $r = 2$.

$$C_2^{10} = \frac{10!}{2! \cdot (10-2)!} = \frac{10!}{2! \cdot 8!} = \frac{10 \cdot 9 \cdot 8!}{2! \cdot 8!} = \frac{10 \cdot 9}{2} = 45$$

Answer: 45 matches.

11. Permutations (order of the sets matter)

A **permutation** is an arrangement of all or part of a set of objects, with regard to the order of the arrangement. This means that ABC is considered a different permutation than CBA.

$$P_r^n = \frac{n!}{(n-r)!}$$

Example: How many different ways can you select two letters from the set of letters: A, B, and C? (Hint: In this problem, order is important; i.e., AB is considered a different arrangement than BA).

Solution: One way to solve this problem is to list all of the possible selections of two letters from the set of A, B, and C. They are: AB, BA, BC, CB, AC and CA. Thus, there are 6 possible combinations.

Another approach is to use permutations. We have 3 distinct objects so $n = 3$. And we want to arrange them in groups of 2, so $r = 2$. Thus, the number of combinations is:

$$P_2^3 = \frac{3!}{(3-2)!} = \frac{3!}{1!} = \frac{1 \cdot 2 \cdot 3}{1} = 6$$

Answer: 6 different ways.

Example: Tanvil forgot his password, but he remembers that his 4-digit password consists of last 5 digits of his social security number, 92693. How many possible combinations are there to guess his password?

Solution: There are four spaces to be filled: _, _, _, _.

The first space can be filled by any one of the five digits. The second space can be filled by any of the remaining four digits. The third space can be filled by any of the three remaining digits and the final space must be filled by any of the two remaining digits. So, $5 \cdot 4 \cdot 3 \cdot 2 = 120$ possibilities. But there is no difference between two given digits 9, so the number of possible passwords is $\frac{120}{2} = 60$.

Another way to solve this problem is to find the total number of possible permutations:

$$\frac{1}{2}P_4^5 = \frac{1}{2} \cdot \frac{5!}{(5-4)!} = \frac{1}{2} \cdot \frac{5!}{1!} = \frac{1}{2} \cdot 5! = \frac{120}{2} = 60$$

Answer: 60 possible passwords.

12. Probability

Many events can't be predicted with total certainty. The best we can say is how likely they are to happen, using the idea of probability.

$$\text{Probability of an event happening} = \frac{\text{Number of favorable outcomes}}{\text{Number of all possible outcomes}}$$

Tips

- The probability of an event can only be between 0 and 1 and can also be written as a percentage.
- The probability of event A is often written as $P(A)$.
- If $P(A) > P(B)$, then event A has a higher chance of occurring than event B.
- If $P(A) = P(B)$, then events A and B are equally likely to occur.

Example: There are 4 blue marbles and 8 black marbles. What is the probability of picking a blue marble?

Solution:

Number of favorable outcomes $= 4$
Number of all possible outcomes $= 4 + 8 = 12$ $\Big\}$ $P = \frac{4}{12} = \frac{1}{3}$

Answer: $\frac{1}{3}$.

13. Addition Rule

Two events are called **mutually exclusive** if they cannot occur at the same time. When two events, A and B, are mutually exclusive, the probability that A or B will occur is

$$P(A \text{ or } B) = P(A) + P(B)$$

Example: A single 6-sided die is rolled. What is the probability of rolling 1 or 6?
Solution:

$P(1) = \frac{1}{6}$
$P(6) = \frac{1}{6}$ $\Big\}$ $P(1 \text{ or } 6) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

Answer: The slope is -2.2.

18. Slope

If the line passes through the points (x_1,y_1) and (x_2,y_2), then the slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example: Find the slope of the line passing through the points $(0,3)$ and $(5,2)$.

Solution:

$$m = \frac{2-3}{5-0} = -\frac{1}{5} = -0.2$$

Answer: $m = -0.2$.

19. Parallel Lines

Parallel lines have the same slope.

Example: Write the equation of the line parallel to the line $y = -3x + 5$ passing through the point $(3, -10)$.

Solution: The slope of the parallel line is the same as the slope of the given line, so $m = -3$.

Use slope intercept form of the equation:

$$y = -3x + b$$

This line passes through the point $(3, -10)$, so its coordinates satisfy the equation:

$$-10 = -3 \cdot 3 + b$$

$$b = -10 + 9 = -1$$

The equation of the parallel line is $y = -3x - 1$.

Answer: $y = -3x - 1$.

20. Perpendicular Lines

If two lines with equations $y = m_1x + b_1$ and $y = m_2x + b_2$ are perpendicular, then

$$m_1 \cdot m_2 = -1$$

Solution: Find the distance between

$$d = \sqrt{(2-(-4))^2 + (y-(-1))^2} = \sqrt{6^2 + (y+1)^2} = \sqrt{36 + y^2 + 2y + 1} = \sqrt{y^2 + 2}$$

Hence,

$$\sqrt{y^2 + 2y + 37} = 10$$

Square this equation:

$$y^2 + 2y + 37 = 100 \Rightarrow y^2 + 2y - 63 = 0$$

Use quadratic formula:

$$y_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-63)}}{2 \cdot 1} = \frac{-2 \pm \sqrt{256}}{2} = \frac{-2 \pm 16}{2} = -9, 7$$

Answer: $(2, -9), (2,7)$.

COORDINATE PLANE

17. Slope Intercept Formula

$$y = mx + b$$

m – slope

b – y-intercept

Example: What is the slope and y-intercept of the line $y = 7x + 3$?

Solution:

Slope $m = 7$

y-intercept $b = 3$ or point $(0,3)$.

Answer: $m = 7, b = 3$.

Example: What is the slope of the line given by the equation $11x + 5y = 32$?

Solution: Rewrite the equation in the slope intercept form:

$$5y = 32 - 11x$$

$$y = \frac{32}{5} - \frac{11}{5}x \Rightarrow y = 6.4 - 2.2x$$

Hence, th

Solution: Find the slope

$$9y = 3x - 5 \Rightarrow y = \frac{1}{3}x - \frac{5}{9}$$

The slope of the given line is $m_1 = \frac{1}{3}$. If the slope of the perpendicular line is m_2, then

$$\frac{1}{3} \cdot m_2 = -1 \Rightarrow m_2 = -3$$

The slope intercept form of the equation of perpendicular line is $y = -3x + b_2$

This line passes through the point $(1, -3)$, thus

$$-3 = -3 \cdot 1 + b_2 \Rightarrow b_2 = 0 \text{ and } y = -3x$$

Answer: $y = -3x$.

21. Reflections

When you reflect a point across the x-axis, the x-coordinate remains the same, but the y-coordinate transformed into its opposite (its sign is changed).

$$(x,y) \rightarrow (x, -y)$$

When you reflect a point across the y-axis, the y-coordinate remains the same, but the x-coordinate transformed into its opposite (its sign is changed).

$$(x,y) \rightarrow (-x,y)$$

When you reflect a point across the origin, the x-coordinate and the y-coordinate is transformed into its opposite (its sign is changed).

$$(x,y) \rightarrow (-x,-y)$$

Example: What is the new coordinates when a point $(1,3)$ is reflected across the x-axis?

Solution:

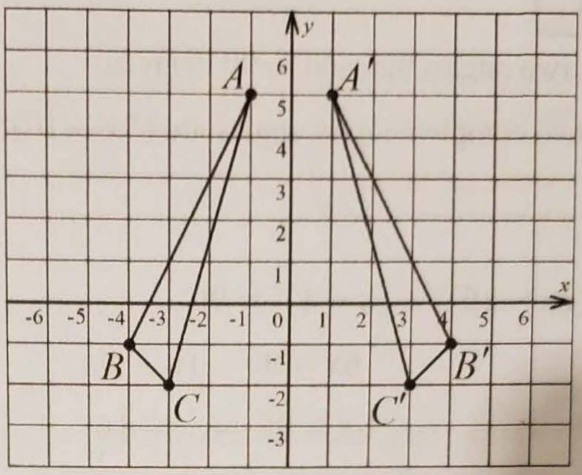
$$(1,3) \rightarrow (1,-3)$$

Answer: $(1, -3)$.

Example: A triangle with vertices $A(-1,5), B(-4,-1)$ and $C(-3,-2)$ is reflected across the x-axis. What are the coordinates of the image triangle?

Solution:

$$A(-1,5) \rightarrow A'(1,5)$$
$$B(-4,-1) \rightarrow B'(4,-1)$$
$$C(-3,-2) \rightarrow C'(3,-2)$$

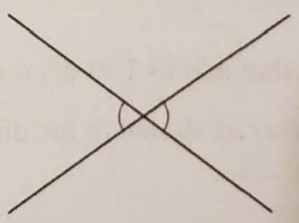


Answer: $(1,5), (4,-1), (3,-2)$.

GEOMETRY

22. Vertical Angles

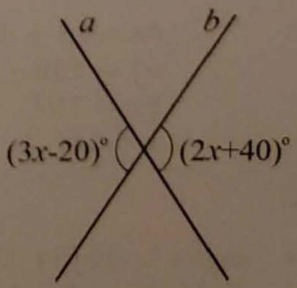
Vertical Angles are the *angles* opposite each other when two lines cross.



A primary property of vertical angles is that they are **congruent**. In other words, they have the same angle measure.

Example: Two lines a and b intersect. If two opposite angles measure $(2x + 40)°$ and $(3x - 20)°$, what is the value of x?

Solution: Angles $(2x + 40)°$ and $(3x - 20)°$ are vertical angles, so



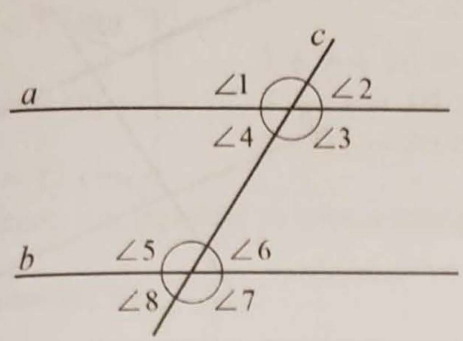
$$2x + 40 = 3x - 20$$

$$40 + 20 = 3x - 2x$$

$$x = 60$$

Answer: $x = 60$.

Parallel lines cut by a transversal



Lines a and b are parallel and they are cut by a line c called **transversal**.

25. Alternate Interior Angles

Alternate interior angles are a pair of angles on the inner side of each of two parallel lines but on opposite sides of the transversal. Alternate interior angles are congruent.

$$\angle 3 \cong \angle 5$$
$$\angle 4 \cong \angle 6$$

26. Alternate Exterior Angles

Alternate exterior angles are a pair of angles on the outer side of each of those two lines but on opposite sides of the transversal. Alternate exterior angles are congruent.

$$\angle 1 \cong \angle 7$$
$$\angle 2 \cong \angle 8$$

27. Corresponding Angles

There are four pairs of corresponding angles. Corresponding angles are congruent.

$$\angle 1 \cong \angle 5$$
$$\angle 2 \cong \angle 6$$
$$\angle 3 \cong \angle 7$$
$$\angle 4 \cong \angle 8$$

28. Same Side Interior Angles

Same side interior angles are a pair of interior angles on the same side of the transversal. Same side interior angles add up to 180°.

$$\angle 3 + \angle 6 = 180^\circ$$
$$\angle 4 + \angle 5 = 180^\circ$$

23. Complementary angles

Complementary angles are two angles that add to 90 degrees.

Example: The measure of two complementary angles are $(5x - 10)°$ and $(x + 4)°$. Find the measure of each angle.

Solution:

Complementary angles $\Rightarrow (5x - 10)° + (x + 4)° = 90°$.

$$6x = 90 + 10 - 4$$

$$6x = 96 \Rightarrow x = 16$$

Hence, $(5x - 10)° = (5 \cdot 16 - 10)° = 70°$

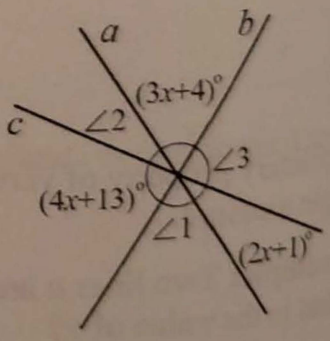
$(x + 4)° = (16 + 4)° = 20°$

Answer: $70°$ and $20°$.

24. Supplementary Angles

Supplementary angles are two angles that add to 180 degrees.

Example: Three lines a, b and c intersect as shown in the diagram. Find the measure of angles $\angle 1$, $\angle 2$ and $\angle 3$.



Solution:

Angles 2 and $(2x + 1)°$ are vertical angles, so

$$\angle 2 = (2x + 1)°$$

Angles $(4x + 13)°$, $(3x + 4)°$ and $\angle 2$ are supplementary angles, so

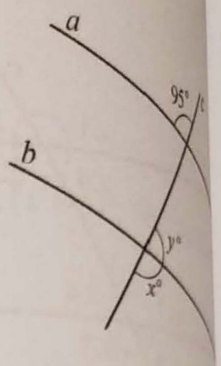
$$4x + 13 + 3x + 4 + 2x + 1 = 180 \Rightarrow 9x + 18 = 180$$
$$9x = 180 - 18$$
$$9x = 162$$
$$x = 18$$

Now,

- angles 1 and $(3x + 4)°$ are vertical $\Rightarrow \angle 1 = (3 \cdot 18 + 4)° = 58°$
- angles 2 and $(2x + 1)°$ are vertical $\Rightarrow \angle 2 = (2 \cdot 18 + 1)° = 37°$
- angles 3 and $(4x + 13)°$ are vertical $\Rightarrow \angle 3 = (4 \cdot 18 + 13)° = 85°$

Answer: $58°, 37°, 85°$.

Example: Two parallel lines a and b are cut by transversal c as shown in the diagram. What are the values of x and y?



Solution: Supplementary angle to angle $y°$ is corresponding angle with $95°$ angle, so

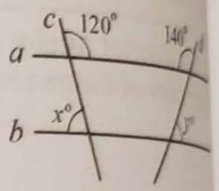
$$y° = 180° - 95° = 85°$$

Angles $x°$ and $y°$ are supplementary, thus

$$x° = 180° - 85° = 95°$$

Answer: $x° = 95°, y° = 85°$.

Example: Two parallel lines a and b are cut by two transversals c and d. What are the values of x and y?



Solution:

Supplementary angle to $120°$ angle is alternate interior angle with $x°$ angle, so $x° = 180° - 120° = 60°$.

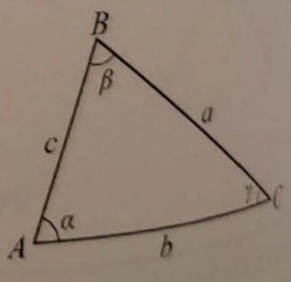
Supplementary angle to $140°$ angle is alternate interior angle with $y°$ angle, so

$$y° = 180° - 140° = 40°.$$

Answer: $x° = 60°, y° = 40°$.

29. Triangle

A **triangle** is a 3-sided figure formed by joining three line segments together at their endpoints.



A triangle has

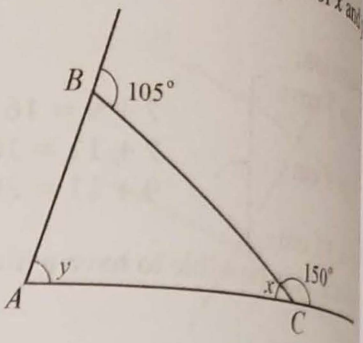
- three sides;
- three vertices;
- three angles.

30. Triangle Inequality Theorem

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

$$a + b > c$$
$$a + c > b$$
$$b + c > a$$

Example: In triangle ABC, two exterior angles have measures $105°$ and $150°$. Find the value of x and y.



Solution:

Angles with measures x and $150°$ are supplementary angles, so

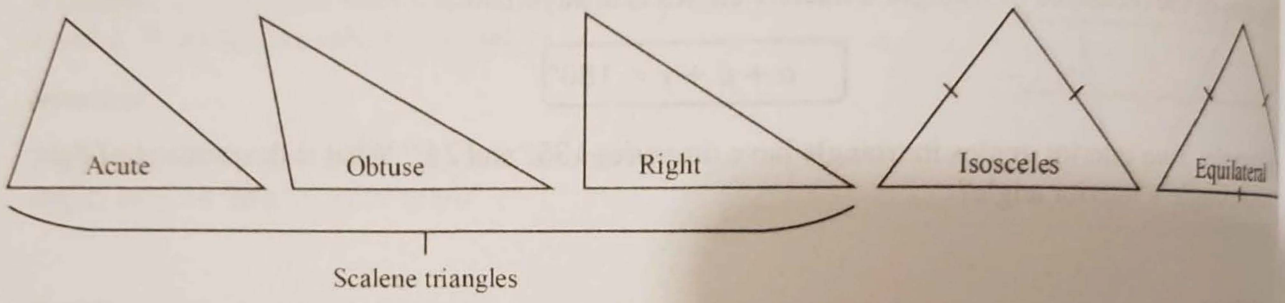
$$x + 150° = 180°$$
$$x = 180° - 150° = 30°$$

By the exterior angle theorem,

$$x + y = 105° \Rightarrow y = 105° - 30° = 75°$$

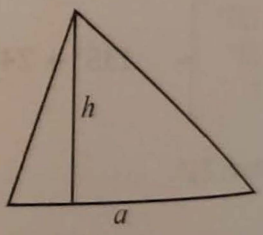
Answer: $x = 30°, y = 75°$.

34. Types of Triangles



35. Area of the Triangle

$$A = \frac{1}{2} \cdot a \cdot h$$



Example: Find the area of an acute triangle with a base of 25 inches and a height of 8 inches.

Solution: Use area formula:

$$A = \frac{1}{2} \cdot 25 \cdot 8 = 100 \text{ in}^2$$

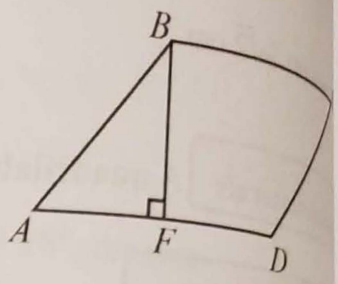
Answer: 100 in^2.

Example: The area of a triangle shaped mat is 36 square feet and the base is 9 feet. Find the height of the mat.

Solution: Use area formula:

$$A = \frac{1}{2} \cdot a \cdot h \Rightarrow 36 = \frac{1}{2} \cdot 9 \cdot h$$

Example: In parallelogram $ABCD$, height BF bisects the side AD and form isosceles right triangle ABF. If the length of AD is 12 cm what is the area of the parallelogram $ABCD$?



Solution: If BF bisects the side AD, then $AF = 5$ cm.

Triangle ABF is isosceles right triangle, then $AF = BF = 5$ cm.

The area of the parallelogram $ABCD$ is $A = AD \cdot BF = 10 \cdot 5 = 50\ \text{cm}^2$.

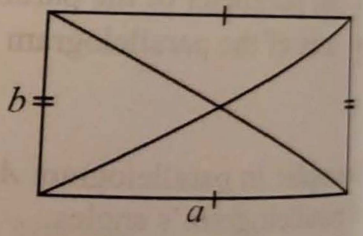
Answer: $50\ \text{cm}^2$.

44. Rectangle

A **rectangle** is a parallelogram with four right angles.

Tips

- opposite side congruent
- all angles are right angles
- diagonals are congruent
- diagonals bisect each other
- the perimeter of the rectangle is $P = 2(a + b)$
- the area of the rectangle is $A = a \cdot b$



Example: The area of the rectangle with one side of 8 in is $48\ \text{in}^2$. What is the length of the diagonal of the rectangle?

Solution: Use the area formula to find the length of the second rectangle's side:

$$48 = 8b \Rightarrow b = \frac{48}{8} = 6\ \text{in}$$

Use Pythagorean theorem for the right triangle formed by two consecutive rectangle's sides and diagonal:

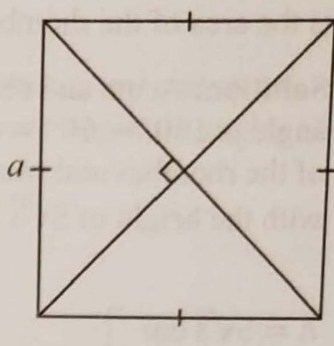
$$d^2 = 6^2 + 8^2 = 36 + 64 = 100 \Rightarrow d = 10\ \text{in}$$

Answer: 10 in

BBY-TARIQ

45. Square

A **square** is a rectangle with all congruent sides.



Tips

- all sides are congruent
- all angles are right angles
- diagonals are congruent
- diagonals bisect each other
- diagonals are perpendicular and splits the square into four congruent right triangles
- the perimeter of the square is $P = 4a$
- the area of the square is $A = a^2$
- the diagonal of the square has length $d = a\sqrt{2}$

Example: What is the area of the square with diagonal of $12\sqrt{2}$ cm.

Solution:

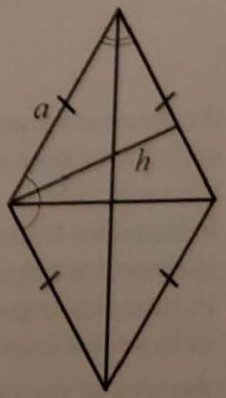
$d = 12\sqrt{2}$ cm
$a = ?$
$A = ?$

$d = a\sqrt{2} \Rightarrow 12\sqrt{2} = a\sqrt{2}$
$a = 12$ cm
$A = a^2 = 12^2 = 144$ cm^2

Answer: 144 cm^2.

46. Rhombus

A **rhombus** is a parallelogram with all congruent sides.



Tips

- all sides congruent
- opposite angles congruent
- two consecutive angles are supplementary
- diagonals are perpendicular
- diagonals are angles bisectors
- diagonals bisect each other
- the perimeter of the rhombus is $P = 4a$
- the area of the rhombus is $A = a \cdot h$

Example: Two angles in trapezoid have measures 46° and 72°. What are the measures of remaining two trapezoid's angles?

Solution: These two angles are not the same side interior angles (their measures do not add up to 180°), so they are adjacent to different trapezoid's legs. Find two supplementary angles to angles of 46° and 72°:

$$180° - 46° = 134°$$

$$180° - 72° = 108°$$

Answer: 134° and 108°.

Example: What is the area of trapezoid with bases of 14 cm and 22 cm and height of 4 cm?

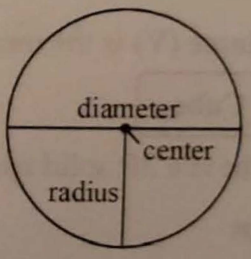
Solution: Use area formula:

$$A = \frac{14 + 22}{2} \cdot 4 = 18 \cdot 4 = 72 \text{ cm}^2$$

Answer: 72 cm^2.

48. Circle

A **circle** is the locus of all points equidistant from a central point.



Tips

- the diameter d is twice the radius r
- the circumference of the circle is $c = 2\pi r = \pi d$
- the area of the circle is $A = \pi r^2$

Example: What is the circumference of the circle with diameter 48 cm?

Solution: Use circumference formula:

$$c = \pi d \Rightarrow c = 48\pi \text{ cm}$$

Answer: 48π cm.

Example: What is the area of the circle with circumference of 12π ft?

Solution:

$c = 12\pi$ ft
$A = ?$

$c = 2\pi r \Rightarrow 2\pi r = 12\pi, r = 6$ ft

$A = \pi r^2 \Rightarrow A = \pi \cdot 6^2 = 36\pi \text{ ft}^2$

Answer: 36π ft^2.

49. Polygons

The sum of the measures of all interior angles of n-sided polygon is

$$(n-2)\cdot 180^\circ$$

Example: What is the measure of one interior angle of regular hexagon?

Solution:

$n = 6 \Rightarrow$ the sum of the measures of all interior angles is $(6-2)\cdot 180^\circ = 720^\circ$.

All interior angles in regular hexagon are congruent, so the measure of each of these angles is

$$\frac{720^\circ}{6} = 120^\circ$$

Answer: 120°.

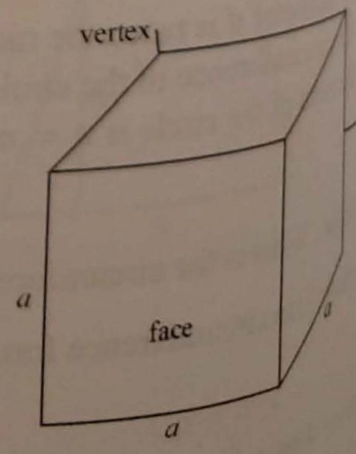
3D Shapes

Surface Area (SA) is the sum of the area of all sides of the shape.

Volume (V) is the measure of the space inside of a solid figure.

50. Cube

Cube is a 3D solid with six square faces.



Tips

- has 6 faces
- has 12 edges
- has 8 vertices
- all faces are congruent squares
- the SA of the cube is $SA = 6a^2$
- the volume of the cube is $V = a^3$
- the diagonal of the cube is $d = a\sqrt{3}$

Example: If the SA of the cube is 216 square centi... of the cube?